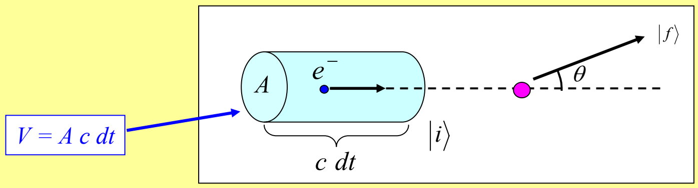
16.451 Lecture 6: Cross Section for Electron Scattering *Sept. 27, 2005* 1

Recall:
A beam particle will scatter from the target particle into solid angle $d\Omega$ at (θ, ϕ) if it approaches within the corresponding area $d\sigma = (d\sigma/d\Omega)\, d\Omega$ centered on the target.



- Electron (speed c) is in a plane wave state normalized in volume V as shown.
- Probability of scattering at angle θ is given by the ratio of areas: $P(\theta) = \dfrac{d\sigma(\theta)}{d\Omega}\,\dfrac{d\Omega}{A}$
- Transition rate λ_{if} = (electrons/Volume) x (Volume/time) x $P(\theta)$

$$\lambda_{if} = \left(\frac{1}{V}\right)\left(\frac{A\,c\,dt}{dt}\right)\left(\frac{d\sigma/d\Omega}{A}\right) d\Omega \;=\; \left(\frac{c}{V}\right)\left(\frac{d\sigma}{d\Omega}\right) d\Omega$$

Recall from last time: 2

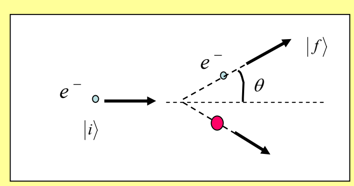
$$\lambda_{if} = \frac{2\pi}{\hbar}\left|M_{if}\right|^2 \rho_f$$

(transitions / sec)

$$\left\{\begin{array}{l} M_{if} = \int \psi_f^*\, V(\vec{r})\, \psi_i\, d^3r \\ \rho_f = dn/dE_f \end{array}\right.$$

And we just found that:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \lambda_{if}\,\frac{V_n}{c\,d\Omega}$$



where V_n is the normalization volume for the plane wave electron states, and λ_{if} is the **transition rate** from the initial to final state, which we calculate using a standard result from quantum mechanics known as "Fermi's Golden Rule:"

We will first calculate the matrix element M_{if} and then the density of states ρ_f

Matrix Element for Scattering, M_{if} (Nonrelativistic QM treatment!!!)

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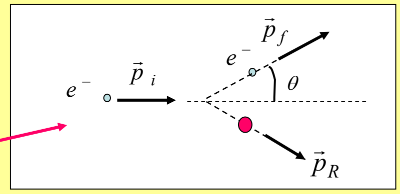
$$M_{if} \equiv \int \psi_f^* \, V(\vec{r}) \, \psi_i \, d^3 r$$

3 – momenta: p_i, p_f, p_R



Use plane wave states to represent the incoming and outgoing electrons, and

let $p_i = \hbar\, k_i,\; p_f = \hbar\, k_f,\; p_R = \hbar\, q,$ and **normalization volume** = V_n

$$\psi_i(\vec{r}) = \frac{1}{\sqrt{V_n}}\, e^{i\vec{k}_i \bullet \vec{r}} \qquad \psi_f(\vec{r}) = \frac{1}{\sqrt{V_n}}\, e^{i\vec{k}_f \bullet \vec{r}}$$

(Note: slight change of notation here from last class to make sure we don't miss any factors of ħ. The recoil momentum of the proton in MeV/c is p_R; the momentum transfer in fm $^{-1}$ is $q = p_R / \hbar$)

Matrix element, continued...

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$$M_{if} \equiv \int \psi_f^* \, V(\vec{r}) \, \psi_i \, d^3 r$$

$$= \frac{1}{V_n} \int e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{r}}\, V(\vec{r})\, d^3 r \;=\; \frac{1}{V_n} \int e^{i\,\vec{q}\,\bullet\,\vec{r}}\, V(\vec{r})\, d^3 r$$

Insight #1: RHS is the Fourier transform of the scattering potential $V(r)$, and it only depends on the momentum transfer q !

Next, proceed with caution:

$V(r)$ is the Coulomb potential of the **extended charge distribution** of the **target atom** that our electron is scattering from ...

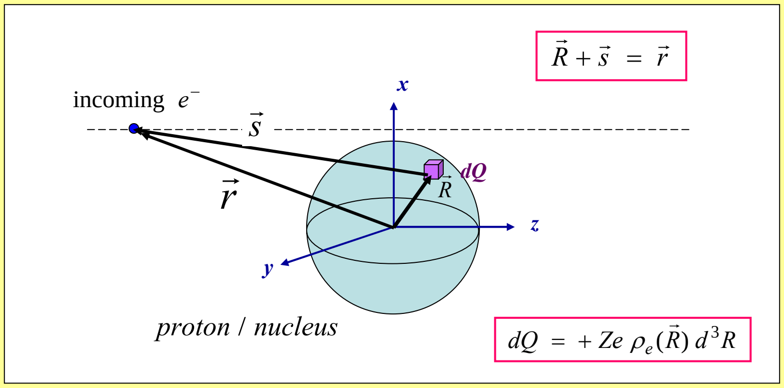
- **at large distances**, the atom is electrically neutral, so $V(r) \to 0$ **faster than 1/r**
- **at short distances**, we have to keep track of geometry carefully, **accounting for the details** of the proton (or nuclear) charge distribution....

Screened Coulomb potential: 5

$$V(r) = -\frac{Z\,e^2}{4\pi\,\varepsilon_o\,r}\,e^{-r/\alpha}$$

For the **atom**, where Z is the atomic number, and α is a distance scale of order Å, the atomic radius.

But the electron interacts with charge elements dQ inside the nucleus:



Bottom line:

$$V(r) = -e\int_{nucleus}\frac{dQ\;e^{-r/\alpha}}{4\pi\,\varepsilon_o\,s}$$

Now do the integral... 6

$$V(r) = -e\int_{nucleus}\frac{dQ\;e^{-r/\alpha}}{4\pi\,\varepsilon_o\,s}$$

$\vec{r}$ = electron coordinate

$\vec{R}$ = coordinate of dQ

$\vec{s}$ = displacement of electron from dQ

$$dQ = +Ze\,\rho_e(\vec{R})\,d^3R$$

$$\int dQ = +Ze$$

normalization: $[\rho] = m^{-3}$

substitute for dQ:

$$V(r) = -\frac{Z\,e^2}{4\pi\,\varepsilon_o}\int_{nucleus}\frac{\rho_e(R)\;e^{-r/\alpha}}{s}\,d^3R$$

Finally, for the matrix element:

$$M_{if} = \frac{1}{V_n}\int e^{i\,\vec{q}\cdot\vec{r}}\,V(\vec{r})\,d^3r = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)\int_{all\ space} d^3r\;e^{i\vec{q}\cdot\vec{r}}\int_{nucleus}\frac{\rho_e(R)\;e^{-r/\alpha}}{s}\,d^3R$$

How to deal with the variables:

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$$M_{if} = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right) \int_{all\ space} d^3r\ e^{i\vec{q}\cdot\vec{r}} \int_{nucleus} \frac{\rho_e(R)\ e^{-r/\alpha}}{s}\ d^3R$$

problem: r, R and s in here!

Solution:

1. inside the nucleus, where $\rho\,(R)$ differs from zero, $e^{-r/\alpha} \cong 1 \cong e^{-s/\alpha}$

   (where the screening factor really matters is at large r, and there r → s to an even better approximation!)

2. there is a one to one mapping between all electron positions r and all displacements from the charge element dQ, so: $\int_{all\ space} d^3r = \int_{all\ space} d^3s$

$$M_{if} = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right) \iint e^{i\vec{q}\cdot\vec{r}}\ \rho_e(R)\ d^3R\ \frac{e^{-s/\alpha}}{s}\ d^3s$$

(this expression can be factored into 2 parts ...)

Finishing the integral for M_{if}:

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$$M_{if} = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right) \iint e^{i\vec{q}\cdot\vec{r}}\ \rho_e(R)\ d^3R\ \frac{e^{-s/\alpha}}{s}\ d^3s$$

use the relation: $\vec{r} = \vec{R} + \vec{s}$ to simplify...

$$M_{if} = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right) \boxed{\int e^{i\vec{q}\cdot\vec{R}}\rho_e(R)\,d^3R} \times \boxed{\int e^{i\vec{q}\cdot\vec{s}}\ \frac{e^{-s/\alpha}}{s}\,d^3s}$$

Fourier transform of the nuclear charge density ≡ $F(q^2)$

Exact integral: $\frac{4\pi}{q^2+\alpha^{-2}}$

(Eureka!) ☺

$$M_{if} = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)\left(\frac{4\pi}{q^2+\alpha^{-2}}\right) F(q^2)$$

What does this mean? 9

$$M_{if} = \text{(constants)} \times \text{(exact integral)} \times \text{(Fourier transform of } \rho(\text{R}))$$

$$M_{if} = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)\left(\frac{4\pi}{q^2+\alpha^{-2}}\right) F(q^2)$$

Consider: $F(q^2) \equiv \int_{\text{all space}} e^{i\vec{q}\cdot\vec{r}}\,\rho_e(r)\,d^3r$

If the scattering object is a **point charge**, $\rho_e(r) = \delta^3(\vec{r})$, i.e. the normalized charge density is a **Dirac delta function**, with the property:

$$\int_{\text{all space}} \delta^3(\vec{r})\; d^3r \equiv 1 \quad \Rightarrow \quad \boxed{F(q^2) = 1 \text{ for a point charge}}$$

(Really important result!)

N.B. for a delta function: $\int f(\vec{r})\,\delta^3(\vec{r})\,d^3r = f(0)$

Finally, work out the density of states factor: 10

from slide 2...

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{2\pi}{\hbar}\,\frac{V_n}{c\,d\Omega}\,\left|M_{if}\right|^2 \rho_f$$

$$M_{if} = \int \psi_f^*\, V(\vec{r})\,\psi_i\; d^3r$$

$$\rho_f = dn/dE_F$$

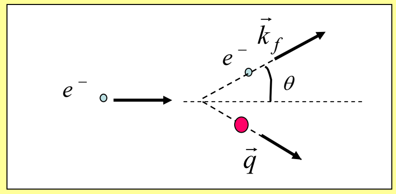
for the cross-section:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{2\pi}{\hbar}\,\frac{1}{c\,V_n}\left(\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)^2\left(\frac{4\pi}{q^2+\alpha^{-2}}\right)^2\left(F(q^2)\right)^2 \boxed{\frac{dn}{dE_F\,d\Omega}}$$

All we have left to calculate is the "density of states" factor, $\frac{dn}{dE_F\,d\Omega}$ where E_F is the total energy in the final state when the electron scatters at angle θ, and this factor accounts for the number of ways it can do that.

Consider the **total** final state energy: 11

$$\frac{dn}{dE_F\, d\Omega} = \frac{dn}{dp_f\, d\Omega}\left(\frac{dp_f}{dE_F}\right)_\theta$$



$$E_F = E' + E_R$$

(electron) (recoil)

$$E_F = (cp_f + mc^2) + (Mc^2 + K)$$

(being careful with the factor of c !)

$$\frac{dp_f}{dE_F} = \frac{W\, p_f}{Mc^3\, p_i} \approx \frac{1}{c}$$

$$dE_F \cong c\, dp_f$$

$$\frac{dn}{dE_F\, d\Omega} = \frac{dn}{c dp_f\, d\Omega}$$

This is useful because the momentum states are quantized – we have our electrons in a normalization volume, and we can "count the states" inside ...

Counting the scattered electron momentum states: 12

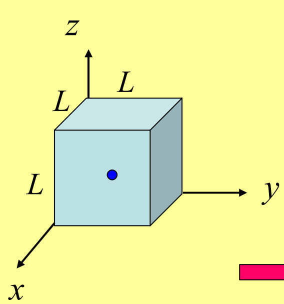
Recall the wave function:

$$\psi_f(\vec{r}) = \frac{1}{\sqrt{V_n}}\, e^{i\vec{k}_f \bullet \vec{r}} \quad with \quad p_f = \hbar k_f$$

The normalization volume is arbitrary, but we have to be consistent ....

let $V_n = L^3$, i.e. the electron wave function is contained in a cubical box,

so its **wave function must be identically zero on all 6 faces of the cube.**



Since: $\vec{k}_f \bullet \vec{r} \equiv k_x x + k_y y + k_z z$

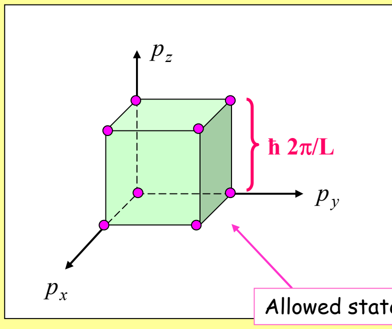
Then it follows that:

$$\psi_f(x,y,z) = \frac{1}{\sqrt{L^3}}\, e^{ik_x x}\, e^{ik_y y}\, e^{ik_z z}$$

$$k_x L = n_x\, 2\pi, \; etc. ...$$

So, momentum is quantized on a 3-d lattice: 13

$$\vec{p}_f = \hbar\,\vec{k}_f = \hbar\left(\frac{2\pi}{L}\right)\left(n_x\,\hat{i} + n_y\,\hat{j} + n_z\,\hat{k}\right)$$
$$n_x = \pm\,(1, 2, 3\,\ldots) \quad etc.$$



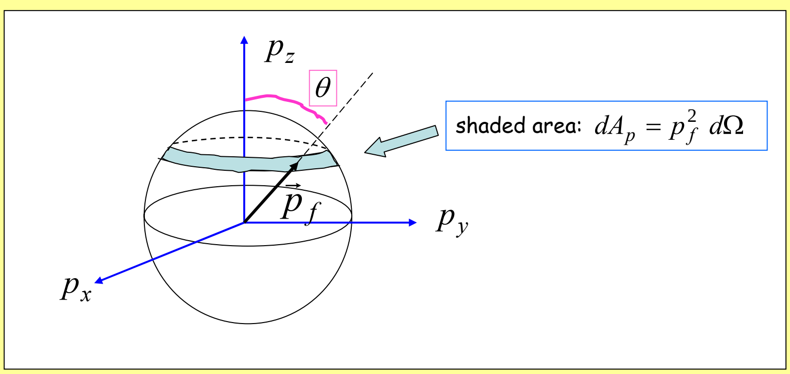
For a relativistic electron beam, the quantum numbers n_x *etc.* are very large, but finite.

We use the quantization relation **not** to calculate the allowed momentum, but rather to calculate **the density of states!**

Allowed states are dots, 1 per cube of volume $\tau_p = (2\pi\hbar/L)^3$

$$\frac{dn}{d\tau_p} = \frac{1 \text{ state}}{(2\pi\,\hbar/L)^3}$$

Finally, consider the scattered momentum into dΩ at θ : 14



number of momentum points in the shaded ring: $dn = \left(\frac{dn}{d\tau_p}\right) \times \left(dA_p\,dp_f\right)$

$$dn = \frac{V_n}{(2\pi\,\hbar)^3}\,p_f^2\,dp_f\,d\Omega$$

End of the calculation: 15

$$dn = \frac{V_n}{(2\pi\hbar)^3}\, p_f^2\, dp_f\, d\Omega$$

We want the density of states factor:

$$\boxed{\frac{dn}{dE_F\, d\Omega} = \frac{dn}{c\,dp_f\, d\Omega} = \frac{V_n}{(2\pi\hbar)^3}\,\frac{p_f^2}{c}}$$

FINALLY, from slide 10:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{2\pi}{\hbar}\,\frac{1}{c\,\cancel{V_n}}\left(\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)^2\left(\frac{4\pi}{q^2+\alpha^{-2}}\right)^2\left(F(q^2)\right)^2\boxed{\left(\frac{\cancel{V_n}}{(2\pi\hbar)^3}\,\frac{p_f^2}{c}\right)}$$

$$= (\text{point charge cross-section}) \times \left(F(q^2)\right)^2$$

Result: Cross section for electron scattering from nuclear charge Z: 16

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{4\,Z^2}{\hbar^2\,(\hbar c)^2}\left(\frac{e^2}{4\pi\,\varepsilon_o}\right)^2\frac{p_f^2}{(q^2+\alpha^{-2})^2}\left(F(q^2)\right)^2$$

$$\cong \boxed{\frac{4\,Z^2}{(\hbar c)^4}\left(\frac{e^2}{4\pi\,\varepsilon_o}\right)^2\frac{(cp_f)^2}{q^4}}\ \boxed{\left(F(q^2)\right)^2}$$

point charge cross-section:
most notably, falls off as q^{-4}
(units should be fm^2)

form factor squared
(dimensionless)

Check units: $[\hbar c] = [e^2/4\pi\varepsilon_o] = \text{MeV.fm};\ [cp] = \text{MeV};\ [q] = \text{fm}^{-1}$

$$\left[\frac{d\sigma}{d\Omega}\right] = \frac{1}{(\text{MeV.fm})^4}\,(\text{MeV.fm})^2\,\frac{(\text{MeV})^2}{\text{fm}^{-4}} = \text{fm}^2$$